Positioning and Active Damping of Flexible Beams

Thomas L. Vincent,* Yeong Ching Lin,† and Shiv P. Joshi‡
University of Arizona, Tucson, Arizona 85721

In this paper, we investigate the design of controllers for the positioning and damping of flexible beams. The approach taken has previously been shown to be effective in the design of controllers for the positioning and active damping of spring-mass systems.[1] The control design is based on open-loop positioning followed by closed-loop control for damping. By so doing, it is possible to avoid a conflicting requirements problem associated with traditional state variable feedback design. The open-loop portion of the control is based on optimal control theory, which allows for control saturation. In particular, during this phase of the control, the time to position is minimized. This results in a bang-bang type of control, which is distributed among some "redundant" controllers to allow for flexible mode suppression. Once the beam has been "positioned," the controller switches to a closed-loop phase. The particular closed-loop control used here is based on energy methods and is not a full state variable feedback design. The method is illustrated using a free-free beam example, and the results are compared with a linear system/quadratic cost control design.

Introduction

MANEUVERING a flexible structure, such as a space telescope, will generally excite unwanted flexible modes in the structure. In the case of a space telescope, maneuvering is required to track a target, and the resulting excitations will cause surface deformations that will degrade reflective beam quality.

In this paper, we examine the problem of designing a control system for a flexible beam that will not only position the beam with flexible mode suppression but will quickly damp out any flexible energy remaining in the beam after the maneuver takes place. The problem as formulated here is nontraditional, since the number and location of actuators are not specified a priori. Even if the location of the actuators were specified, traditional state variable feedback methods such as linear quadratic (LQ) methods[2-5] or eigenvalue placement methods[6,7] may not yield a satisfactory control design. There are several reasons why one should seek a control design other than state variable feedback. Because of the nature of the system, a large number of state equations will generally be required to adequately model the system. Thus, most of the state variables will not be measured, and a large-order observer will be required in order to implement traditional state variable feedback. There are methods available to deal with this difficulty[2,8,9-11]; however, this is not the only problem with state variable feedback. Our objective is to design a controller that will rapidly position a flexible beam and yet be able to damp out internal energy. Under state variable feedback, these two requirements are in conflict. In order to obtain quick positioning, we need a "stiff" controller, one whose eigenvalues have large negative real parts. In order to do the damping, the controller needs to have a "softer" touch in order to avoid chattering. This requires that the eigenvalues have negative real parts closer to the imaginary axis. Thus, any state variable feedback design will be a compromise between quick positioning and good damping characteristics.

In this paper, we investigate an alternate approach to the control design. As mentioned earlier, since actuator placement is generally an open design question, we wish to keep the placement problem as part of the overall control design considerations. We also want to be able to cope with finite actuator force limits, a problem not dealt with using LQ methods, yet a prevalent one due to the stiff response requirements just mentioned.

The approach taken here is to design a controller that uses open-loop positioning followed by closed-loop control for damping. By doing so, we can avoid the conflicting requirements problem associated with the LQ design. The open-loop portion of the control is based on optimal control theory, which allows for control saturation. In particular, during this phase of the control, the time to position may be minimized. For the problems to be considered here, this results in a bang-bang type of control. Once the structure has been positioned, the controller switches to a closed-loop phase. The particular closed-loop control used here is based on energy methods and is not a full state variable feedback design.

The open-loop approach to positioning the structure allows for further refinements associated with the actuator placement problem. Our objective is to position the structure as quickly as possible and, at the same time, minimize the amount of energy that will go into the flexible modes of the structure.

Since a flexible structure is a continuous system, it will have an infinite number of flexible modes. By modeling the beam in terms of a finite number of differential equations, we will be approximating the continuous system in terms of a lumped system with a finite number of modes.[12-14] The degree of accuracy depends on the order of the lumped system.[15,16] Suppose, for example, the lumped system has r rigid body modes and f flexible body modes. Then, in order to arbitrarily position the beam, at least r actuators are needed. With only r actuators used to position the body, there will be some spillover into the flexible energy in the system. In order to avoid this with the $r+f$-degree-of-freedom lumped system, one could use $r+f$ actuators so as to move the system model as a rigid body; hence, no energy would be imparted into flexible energy. Since the system model becomes more accurate as $f \to \infty$, this would be an impractical way of removing all of the flexible energy. However, this procedure does suggest a practical way of eliminating a large portion of the imparted flexible energy associated with positioning the beam. In particular, we will demonstrate here that, by using $r+q$ actuators

Received June 3, 1988; revision received Feb. 9, 1989.

*Professor, Department of Aerospace and Mechanical Engineering.
†Graduate Student, Department of Aerospace and Mechanical Engineering.
‡Assistant Professor, Department of Aerospace and Mechanical Engineering; currently, Department of Aerospace Engineering, University of Texas at Arlington.

(q actuators are redundant), it is possible to suppress q flexible body modes. We develop a procedure that may be used to select actuator location depending on which flexible body modes are to be suppressed.

After developing the method, we will demonstrate it using a free flexible beam and compare the approach to an LQ control design.

Theoretical Development

According to elementary beam theory (Euler-Bernoulli), the continuous system representation of a beam is[17]

$$m(x)\frac{\partial^2 y(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=f(x,t)$$

$$0<x<L \tag{1}$$

with appropriate initial and boundary conditions. The first term represents inertial forces, the second term represents internal resultant forces due to flexural stiffness, and the term on the right-hand side represents external forces, including control actions. In general, the closed-form solutions to continuous formulation are not available for complex loading and boundary conditions.

A continuous structure with infinite degrees of freedom is represented by finite degrees of freedom in various ways. When the motion of the beam is defined in terms of finite points in the structure, the formulation is known as a finite-element method. The finite-element method discretizes a structure into a finite number of elements and expresses the displacement at any point of the continuous element in terms of a finite number of displacements at the boundaries of the element:

$$w(x,t)=\sum_{i=1}^{N}\phi_i(x)u_i(t) \tag{2}$$

where $w(x,t)$ is the displacement at a point in the element, $u_i(t)$ are the generalized displacements at the nodes, and $\phi_i(x)$ are the shape functions. Substituting Eq. (2) into the kinetic and potential energy expressions, we obtain the following forms:

$$T(t)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij}\dot{u}_i(t)\dot{u}_j(t) \tag{3}$$

and

$$V(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}u_i(t)u_j(t) \tag{4}$$

Substituting Eqs. (3) and (4) into Lagrange's equations of motion, we obtain

$$M\ddot{Y}+KY=F \tag{5}$$

where M and K are mass and stiffness matrices, respectively, F is a vector of generalized nodal forces, and

$$Y=\begin{bmatrix} y_1\\ \theta_1\\ \vdots\\ y_n\\ \theta_n\end{bmatrix}$$

where y_i is the transverse displacement at each node point i, and θ_i is the slope of the neutral axis at node point i. The model as given by Eq. (5) restricts y_i to small displacements and θ_i to small slopes.

The shape functions for the beam element are given as follows (transverse deflection and rotation are the degrees of freedom at each node):

$$\phi_1(x)=1-3\frac{x^2}{L^2}+2\frac{x^3}{L^3}$$

$$\phi_2(x)=x-2\frac{x^2}{L}+\frac{x^3}{L^2}$$

$$\phi_3(x)=3\frac{x^2}{L^2}-2\frac{x^3}{L^3}$$

$$\phi_4(x)=-\frac{x^2}{L}+\frac{x^3}{L^2}$$

The stiffness and mass matrices are as follows:

$$[K]=\frac{EI}{L}\begin{bmatrix} 12/L^2 & -6/L & -12/L^2 & -6/L\\ -6/L & 4 & 6/L & 2\\ -12/L^2 & 6/L & 12/L^2 & 6/L\\ -6/L & 2 & 6/L & 4\end{bmatrix}$$

$$[M]=\frac{\rho AL}{420}\begin{bmatrix} 156 & -22L & 54 & 13L\\ -22L & 4L^2 & -13L & -3L^2\\ 54 & -13L & 156 & 22L\\ 13L & -3L^2 & 22L & 4L^2\end{bmatrix}$$

Equation (5) is a set of second-order differential equations. Control inputs to the system will be by means of actuators located on the various nodes. A diagonal actuator *placement matrix B*

$$B=\begin{bmatrix} b_1 & & \cdots & 0\\ \vdots & b_2 & \ddots & \vdots\\ 0 & & \cdots & b_{2n}\end{bmatrix}$$

is used to specific whether a force or torque actuator is located on a particular node or not. In particular, b_i will have a value of 0 or 1. If $b_i=0$ (i = odd, even), then there is no (force, torque) actuator on node n_j [$j=(i+1)/2$, $j=i/2$]. If $b_i=1$ (i = odd, even), then a (force, torque) actuator is located on node n_j [$j=(i+1)/2$, $j=i/2$]. The control design process that follows will be used to determine the values for b_i. The actuators are assumed to produce generalized forces. Let $u=[u_1, u_2,\ldots, u_{2n}]^T$ be the vector of such forces (i.e., u_1 is force at node 1, u_2 is moment at node 1, etc.). These forces and moments are assumed to be subject to control bounds of the form

$$|u_i|\le U_i, \qquad i=1,\ldots,2n$$

where U_i, $i=1,\ldots,2n$ are fixed constants. It follows that

$$F=Bu \tag{6}$$

Equation (5) may be reduced to a space representation by choosing

$$X_1=Y=\begin{Bmatrix} y_1\\ \theta_1\\ \vdots\\ y_n\\ \theta_n\end{Bmatrix}, \qquad X_2=\dot{Y}=\begin{Bmatrix} \dot{y}_1\\ \dot{\theta}_1\\ \vdots\\ \dot{y}_n\\ \dot{\theta}_n\end{Bmatrix} \tag{7}$$

It follows from Eqs. (5-7) that

$$\dot{X}_1=\dot{Y}=X_2 \tag{8}$$

$$\dot{X}_2=\ddot{Y}=-M^{-1}KX_1+M^{-1}Bu \tag{9}$$

Equations (8) and (9) are rewritten as

$$\dot{X} = \begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ M^{-1}B \end{Bmatrix} u \quad (10)$$

Equation (10) is in state space form, where u is the vector of external forces and $X = [X_1, X_2]^T$ is the vector of state variables.

Another way of choosing state variables is to let

$$M\dot{Y} = P \quad (11)$$

then

$$\dot{P} = M\ddot{Y} = -KY + Bu \quad (12)$$

and

$$\dot{Y} = M^{-1}P \quad (13)$$

Hence, we have

$$\dot{X} = \begin{Bmatrix} \dot{Y} \\ \dot{P} \end{Bmatrix} = \begin{bmatrix} 0 & M^{-1} \\ -K & 0 \end{bmatrix} \begin{Bmatrix} Y \\ P \end{Bmatrix} + \begin{Bmatrix} 0 \\ B \end{Bmatrix} u \quad (14)$$

Equation (14) is an alternative state space representation.

Using standard modal analysis techniques,[12,14,15] we may decouple Eq. (5) (with F replaced by Bu) by using a transformation matrix Φ whose columns consist of system eigenvectors. The decoupled system is of the form

$$\bar{M}\ddot{z} + \bar{K}z = \bar{B}u \quad (15)$$

where

$$Y = \Phi z \quad (16)$$

is a new set of decoupled generalized coordinates,

$$\bar{M} = \Phi^T M \Phi$$

$$\bar{K} = \Phi^T K \Phi$$

are the diagonal mass and stiffness matrices for the decoupled system, and

$$\bar{B} = \Phi^T B$$

relates the actuator inputs to the decoupled system.

We may now outline a procedure for designing the open-loop positioning controller. First of all, the initial and final rigid-body positionings of the system must be specified in terms of the original coordinates y. The initial and final values for the modal coordinates z are then obtained from Eq. (16). We must now choose forces to obtain the desired positioning. One approach consistent with our original objective of quickly positioning the structure would be to use optimal control theory to find forces that will minimize the time required to achieve the desired positioning. In general, this will result in a bang-bang control law.[18,19]

The basic idea for designing the open-loop controller is to first obtain an open-loop control law that would properly position the structure as if it were indeed a rigid body. One then decides how many additional actuators to place on the structure depending on how many flexible modes are to be actively suppressed during the positioning phase of control.

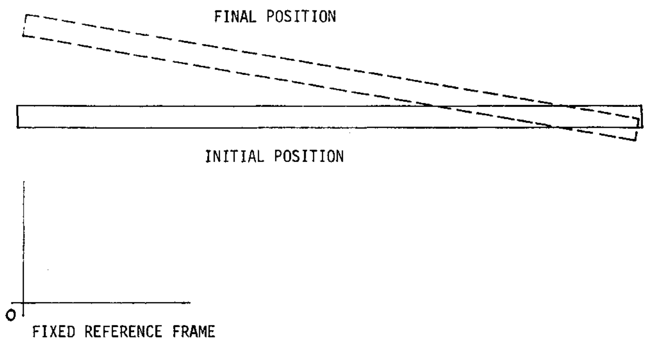

Fig. 1 The UA beam.

Of course, unless $2n$ actuators are used, the modeled system will have some flexible body energy remaining after the positioning takes place. This is due to excitation of unsuppressed modes. It follows that positioning a real structure will always result in some residual flexible body energy. Thus, some additional closed-loop control must be implemented after the open-loop positioning phase. Equations of the form of Eq. (15) suggest that an effective closed-loop controller would incorporate negative velocity feedback for each actuator,[2,8-10,20] with as many actuators also using negative position feedback as there are rigid-body degrees of freedom.

Free Beam Example

Consider the uniform flexible beam illustrated in Fig. 1. It is assumed that there are no forces acting on the beam other than those that will be imposed by means of actuators. The beam is aluminum, 45 in. in length, with a cross section of 0.25 × 1.5 in. The mass and stiffness distributions are $M(x) = 6.76392 \times 10^{-4}$ lb · s²/in.² and $EI(x) = 1.9336 \times 10^4$ lb-in.², respectively. Henceforth, this particular beam will be referred to as the "UA beam."

Our objective is to design a controller for the beam that will move the beam from its initial position to a final position as quickly as possible. The beam will not be considered to be in its final position until it has attained this position with negligible elastic energy.[1] To make the example specific, we will assume the final position to be specified by an upward displacement of the center of mass by 1 in. and a rotation of the beam about the center of mass by 10 deg.

Using a finite-element analysis, one may obtain M, K, and B matrices so that the system may be modeled as equations of the form of Eq. (5). The dimension of the system depends on the number of elements used in the finite-element analysis. For example, by using two equally spaced elements, we obtain the following matrices:

$$M = 10^{-2} \times \begin{bmatrix} 0.1070 & & & \cdots & & 0 \\ \cdot & 0.2579 & & & & \cdot \\ \cdot & & 0.2140 & & & \cdot \\ \cdot & & & 0.5159 & & \cdot \\ \cdot & & & & 0.1070 & \cdot \\ 0 & & & \cdots & & 0.2579 \end{bmatrix}$$

$$K = 10^3 \times \begin{bmatrix} 0.0204 & -0.2292 & -0.0204 & -0.2292 & 0.0000 & 0.0000 \\ -0.2292 & 3.4875 & 0.2292 & 1.7188 & 0.0000 & 0.0000 \\ -0.0204 & 0.2292 & 0.0407 & 0.0000 & -0.0204 & -0.2292 \\ -0.2292 & 1.7188 & 0.0000 & 6.8750 & 0.2292 & 1.7188 \\ 0.0000 & 0.0000 & -0.0204 & 0.2292 & 0.0204 & 0.2292 \\ 0.0000 & 0.0000 & -0.2292 & 1.7188 & 0.2292 & 3.4375 \end{bmatrix}$$

The frequencies for this system are

$$\lambda_1^2 = 0$$
$$\lambda_2^2 = 0$$
$$\lambda_3^2 = 9{,}318$$
$$\lambda_4^2 = 666{,}339$$
$$\lambda_5^2 = 1{,}361{,}437$$
$$\lambda_6^2 = 2{,}018{,}057$$

from which we obtain the modal vectors that, in turn, yield the transform matrix

$$\Phi^T = \begin{bmatrix} 1.0000 & 0.0000 & 1.0000 & 0.0000 & 1.0000 & 0.0000 \\ 1.0000 & 0.0444 & 0.0000 & 0.0444 & -1.0000 & 0.0444 \\ 1.0000 & 0.1343 & -1.0000 & 0.0000 & 1.0000 & -0.1343 \\ 0.0000 & 1.0000 & 0.0000 & -1.0000 & 0.0000 & 1.0000 \\ 0.1619 & -1.0000 & -0.1619 & 0.0000 & 0.1619 & 1.0000 \\ -0.2143 & 1.0000 & 0.0000 & 1.0000 & 0.2143 & 1.0000 \end{bmatrix}$$

From Eq. (15), we obtain

$$\bar{M} = 10^{-2} \times \begin{bmatrix} 0.4280 & & & \cdots & & 0 \\ . & 0.2160 & & & & . \\ . & & 0.4373 & & & . \\ . & & & 0.3176 & & . \\ . & & & & 0.5271 & . \\ 0 & & & \cdots & & 1.0415 \end{bmatrix}$$

$$\bar{K} = 10^{4} \times \begin{bmatrix} 0 & & & \cdots & & 0 \\ . & 0 & & & & . \\ . & & 0.0041 & & & . \\ . & & & 0.6875 & & . \\ . & & & & 0.7176 & . \\ 0 & & & \cdots & & 2.1020 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} b_1 & 0 & b_3 & 0 & b_5 & 0 \\ b_1 & 0.0444b_2 & 0 & 0.0444b_4 & -b_5 & 0.0444b_6 \\ b_1 & 0.1343b_2 & -b_3 & 0 & b_5 & -0.1343b_6 \\ 0 & b_2 & 0 & -b_4 & 0 & b_6 \\ 0.1619b_1 & -b_2 & -0.1619b_3 & 0 & 0.1619b_5 & b_6 \\ -0.2143b_1 & b_2 & 0 & b_4 & 0.2143b_5 & b_6 \end{bmatrix}$$

This system has two rigid-body modes and four elastic-body modes. Clearly, the number of elastic modes will depend on the number of elements used in the finite-element analysis. In the following, we will examine the motion of the UA beam also using a four-element (2 rigid + 8 elastic modes) and a six-element (2 rigid + 12 elastic modes) finite-element analysis. The actual M, k, Φ^T, $\bar{M}$, and $\bar{K}$ matrices for the four-element case are given in the Appendix.

In the two-element beam case, there are three possible actuator locations (each end of the beam plus the middle) with two types of actuators at each location (force and/or torque). For example, for the beam illustrated in Fig. 1, if $b_1 = 1$ and $b_2 = 1$, then both a force and torque actuator would be located at the left end of the beam. That is, b_1, b_3, and b_5 correspond to force actuators at the left, middle, and right ends of the beam, whereas b_2, b_4, and b_6 correspond to torque actuators at the left, middle, and right ends of the beam. We will restrict ourselves to force actuators ($b_2 = b_4 = b_6 = 0$), and we see from $\bar{B}$ that two are required to obtain the required positioning. For example, if we place force actuators at each end of the beam ($b_1 = b_5 = 1$, $b_3 = 0$), the first two decoupled rigid-body modes are given by

$$\bar{m}_1\ddot{z}_1 = u_1 + u_5 \tag{17a}$$

$$\bar{m}_2\ddot{z}_2 = u_1 - u_5 \tag{17b}$$

We obtain the open-loop control law for the actuators by noting that, for a rigid beam, the corresponding translational and rotation motion are obtained from

$$ma = F_t = u_1 + u_5 \tag{18a}$$

$$I\ddot{\theta} = F_r l = (u_1 - u_5)l \tag{18b}$$

where m is the total mass of the beam, a is the vertical upward acceleration, I is the moment of inertia of the beam about the center of mass, and l is the half length of the beam.

More specifically, suppose we wish to obtain the upward displacement of 1 in. with a 10-deg rotation in minimum time with each actuator capable of a given maximum force. For the UA beam, $E = 9.9 \times 10^6$ psi, $\rho = 0.098$ lb/in.^{3}, and $2l = 45$ in. Thus, from Eq. (18) we obtain

$$\begin{matrix} u_1 + u_5 = 6.848 \\ u_1 - u_5 = 8.973 \end{matrix} \Rightarrow \left.\begin{matrix} u_1 = 7.9105 \\ u_5 = -1.0625 \end{matrix}\right\} \quad 0 \leq t \leq 0.025 \tag{19}$$

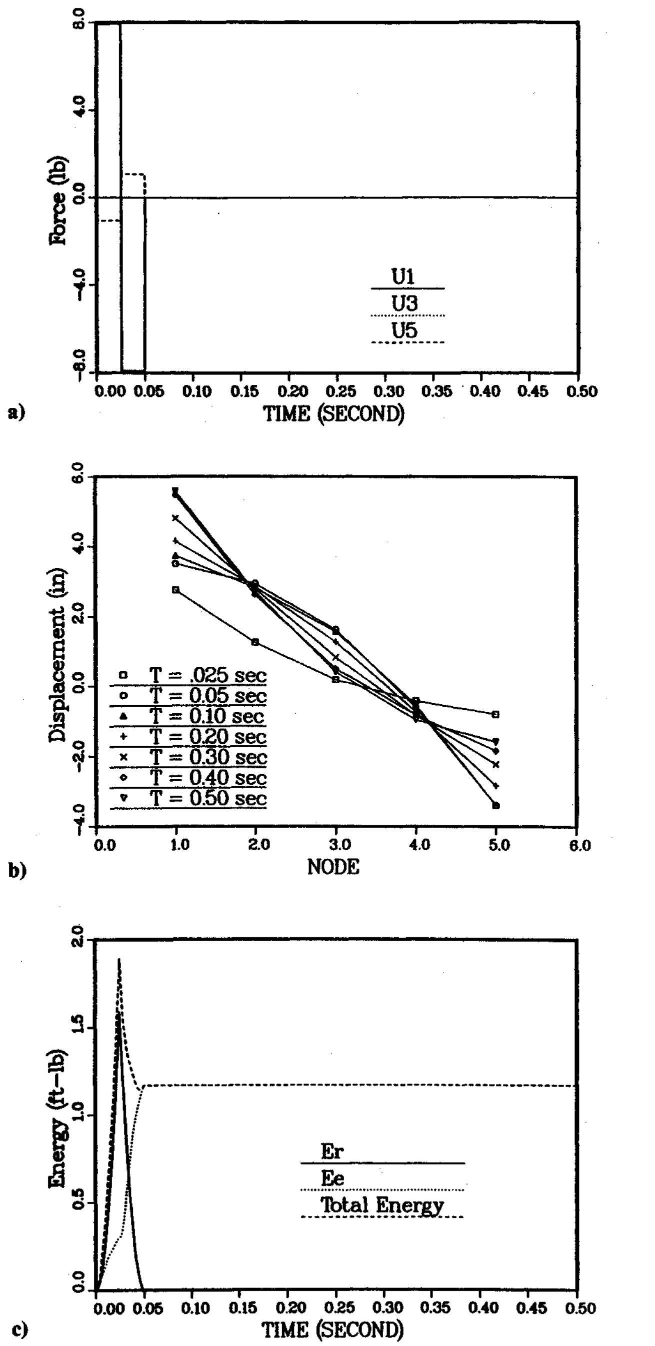

Fig. 2 Open-loop rigid-body positioning of four-element beam with $b_1 = b_5 = 1$, $b_3 = 0$.

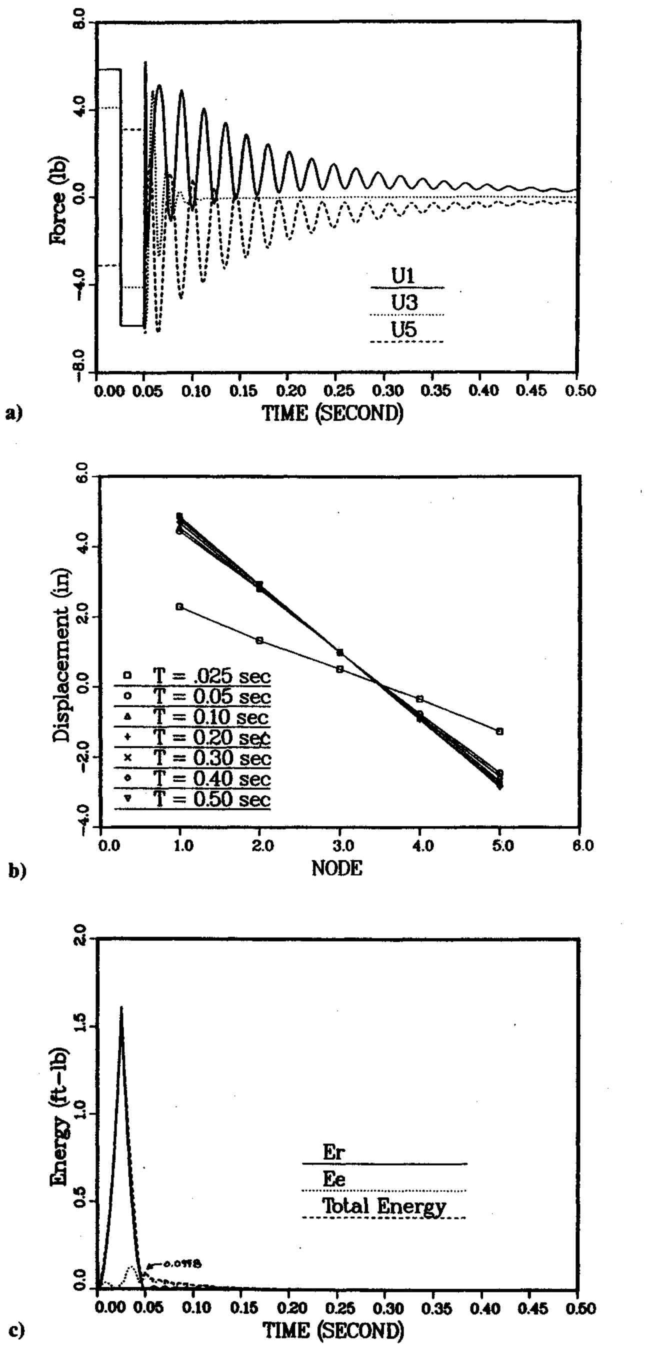

Fig. 3 Open-loop rigid-body positioning of four-element beam with first-mode suppression ($b_1 = b_3 = b_5 = 1$) followed by closed-loop control.

with the controls changing sign over the interval $0.025 < t \leq 0.05$.

We will now temporarily abandon the two-element beam (which favors simplicity) in order to examine the control of a four-element beam (which favors more accuracy). The result of applying Eq. (19) to a four-element beam model is illustrated in Fig. 2a, with the corresponding displacement depicted in Fig. 2b and energy distribution depicted in Fig. 2c. The elastic energy imparted to the beam is greater than one-half the maximum energy input.

Most of the elastic energy of Fig. 2c is contained in the first flexible mode. To suppress the energy going into this mode, we locate a third actuator at node 3 ($b_1 = b_3 = b_5 = 1$). According to the Φ^T matrix for the four-element beam as given in the Appendix, F_t and F_r in Eq. (18) become

$$F_t = u_1 + u_3 + u_5 = 6.848 \tag{20a}$$

$$F_r = u_1 - u_5 = 8.973 \tag{20b}$$

with the additional requirement

$$u_1 - 0.67u_3 + u_5 = 0 \tag{21}$$

to suppress the first flexible mode. From these equations, we obtain the following control forces over the interval $0 \leq t \leq 0.025$:

$$u_1 = 5.8602 \tag{22a}$$

$$u_3 = 4.1006 \tag{22b}$$

$$u_5 = -3.1128 \tag{22c}$$

with a change in sign over the interval $0.025 < t \leq 0.05$. If we follow this open-loop positioning with a closed-loop feedback control of the form[20]

$$u_1 = -\dot{y}_1 - 5(y_1 - 4.9) \tag{23a}$$

$$u_3 = -\dot{y}_3 \tag{23b}$$

$$u_5 = -\dot{y}_5 - 5(y_5 + 2.9) \tag{23c}$$

where $y_1 = 4.9$ and $y_5 = -2.9$ represent the desired final positions for these two coordinates, then the results of Figs. 3a-3c are obtained. Note from Fig. 3c that the elastic energy in the system at the end of 0.05 s has been substantially reduced using mode suppression over that obtained in Fig. 2c. Most of the remaining elastic energy is damped out after about 0.2 s.

Controller Robustness

Since the system model is dependent on the number of elements used in the finite-element analysis, it is of interest to examine the application of a controller based on a given number of elements to a beam modeled by a higher number of elements. We will do so here by first designing a controller for the two-element beam previously discussed. We see from the $\bar{B}$ matrix for the two-element beam that if we use mode suppression on the first flexible mode, then there will be no energy supplied to the second flexible mode (in fact, under force actuator control, this mode is not controllable) and a small amount supplied to the third flexible mode. (The second flexible mode is an antisymmetric mode. The three points at which the actuators are placed are stationary points for this mode.) The open-loop control for this case is obtained from

$$u_1 + u_3 + u_5 = 6.848 \tag{24a}$$

$$u_1 - u_5 = 8.973 \tag{24b}$$

$$u_1 - u_3 + u_5 = 0 \tag{24c}$$

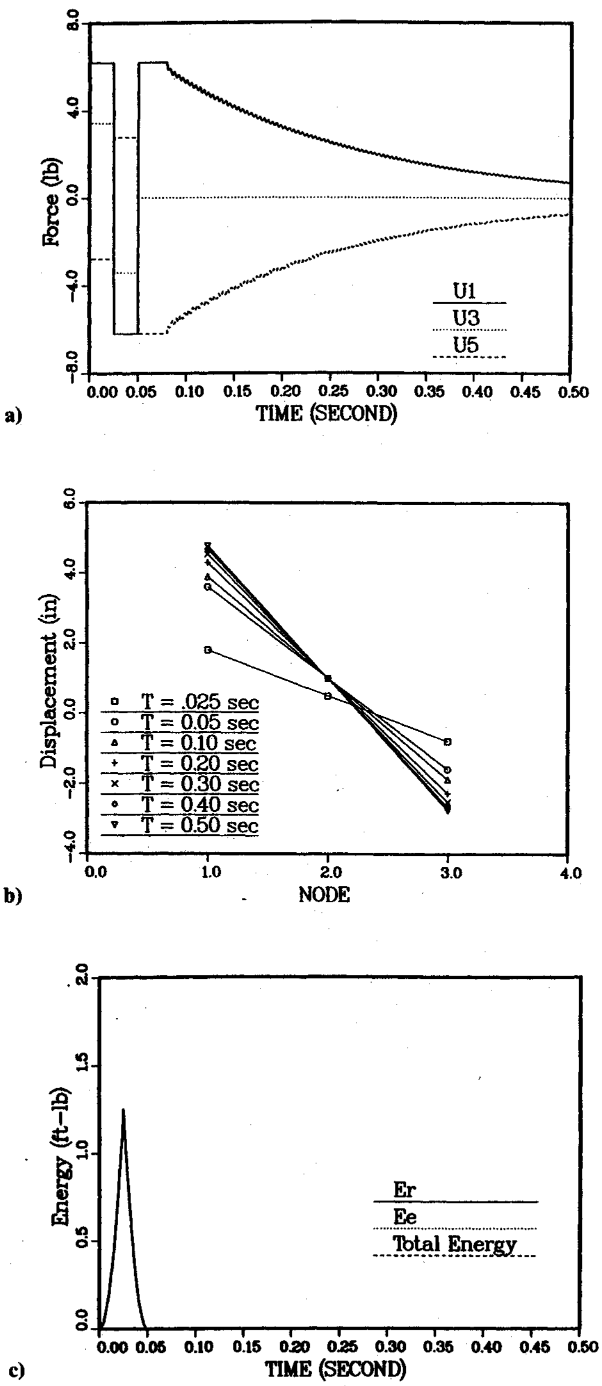

Fig. 4 Open-loop rigid-body positioning of two-element beam with first-mode suppression ($b_1 = b_3 = b_5 = 1$) followed by closed-loop control.

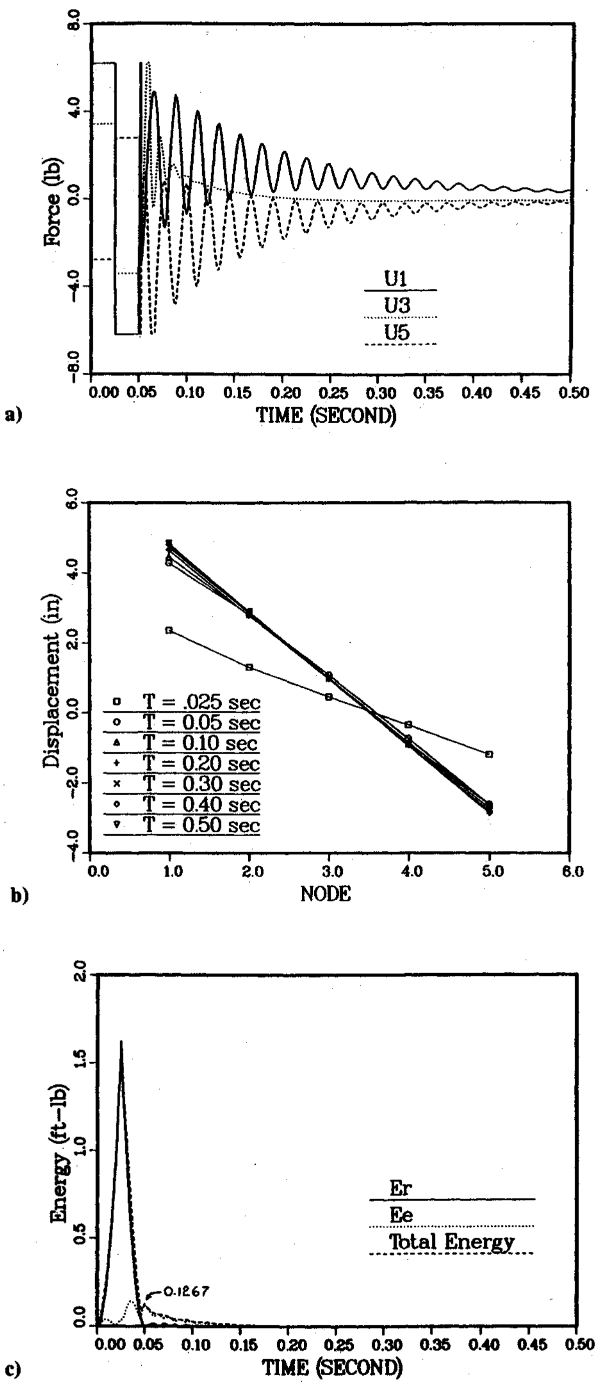

Fig. 5 Response of four-element beam using two-element controller of Fig. 4.

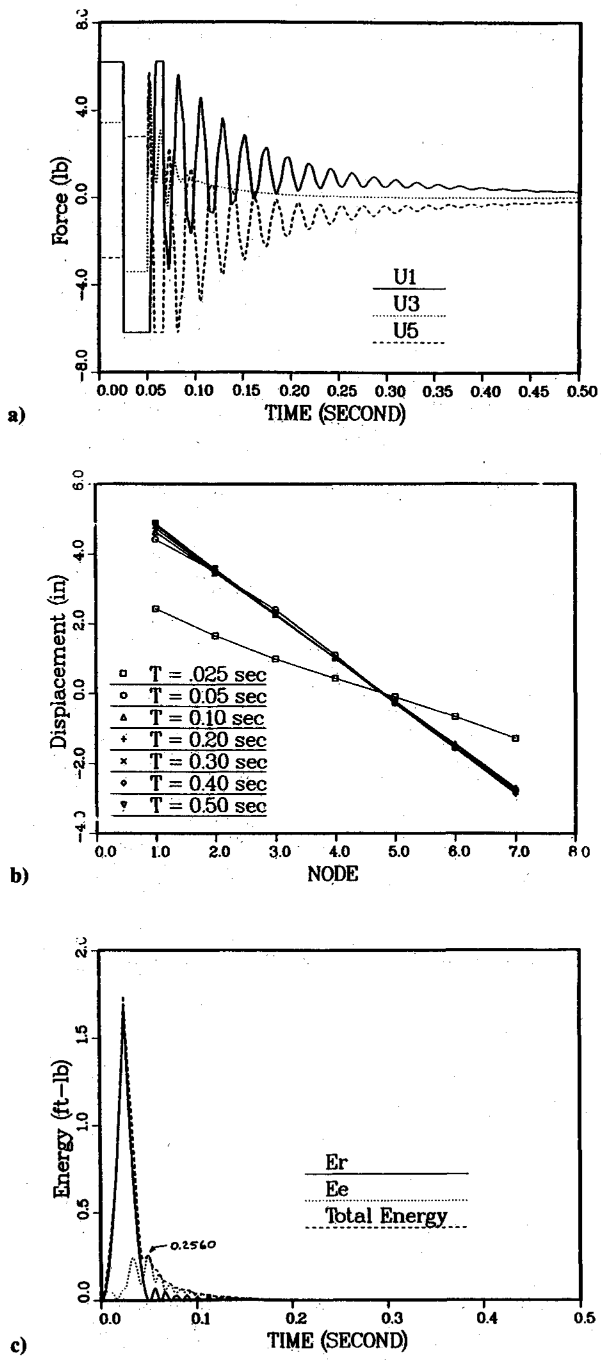

Fig. 6 Response of six-element beam using two-element controller of Fig. 4.

yielding the control law

$$\left.\begin{aligned} u_1 &= 6.200 \\ u_2 &= 3.424 \\ u_3 &= -2.775 \end{aligned}\right\}, \qquad 0 \le t \le 0.025 \tag{25}$$

with $u_i = -u_i$, $i = 1,2,3$ for $0.025 < t \le 0.05$. The same closed-loop controller as that in Eqs. (23) is used after the open-loop positioning. The results are illustrated in Figs. 4a–4c. Note from Fig. 4c that the amount of flexible energy in the beam at the end of the open-loop phase is so small that it does not register on this scale. However, some control is required for final positioning, as indicated in Fig. 4a.

Figures 5a–5c illustrate the response of the four-element UA beam using the two-element open-loop control [Eq. (25)] followed by the closed-loop control [Eqs. (23)]. We note from Fig. 5c that there is an increase in flexible energy at 0.05 s compared with that obtained in Figs. 3c and 4c. This is, of course, to be expected. The third flexible mode frequency obtained by using the four-element beam model is more accurate than that obtained using the two-element model. The two-element model is stiff; therefore, no energy is imparted to the third flexible mode. The controllers uses to generate Figs. 3c and 4c were each designed to eliminate the first flexible mode of the four-element and two-element beam, respectively.

Figures 6a–6c illustrate the response of a six-element UA beam using the two-element open-loop controller [Eq. (25)] followed by the closed-loop control [Eqs. (23)]. Here, again, there is an increase in the flexible internal energy at 0.05 s, as noted in Fig. 6c. As we add more and more elements to the beam, we add additional modes to which energy may be supplied. However, additional increases will be small, since most of the power supplied by the open-loop control would be to the first flexible mode.

LQ Controller Design

We will now examine the robustness of an LQ design for the two-element beam as applied to a four-element beam. However, since the two-element beam under force actuator input is not controllable, we will modify the finite-element analysis somewhat to obtain complete controllability. Rather than divide the beam into two equal elements, we chose the left element to be shorter (22 in.) than the right element (23 in.). The M and K matrices for this case are given in the Appendix. Note that the open-loop positioning/closed-loop damping method of the previous section does not require the complete controllability condition to be satisfied. Since the system is now controllable, we will be able to have arbitrary eigenvalue placement using state variable feedback. Consider first using three actuators ($b_1 = b_3 = b_5 = 1$) to minimize the quadratic performance index

$$J = \int \left\{ x^T \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} x + u^T[R]u \right\} \mathrm{d}t \tag{26}$$

with

$$R = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \tag{27}$$

The first term in Eq. (26) corresponds to the total system energy, and the second term is chosen to penalize control action to the rigid-body modes, as well as control action, which goes into the first flexible mode. The matrix R is obtained by noting that, according to the two-unequal-element-Φ^T matrix as given in the Appendix, we have

$$F_t = u_1 + 0.98u_3 + 0.96u_5$$
$$F_r = 0.98u_1 + 0.01u_3 - u_5$$
$$f_1 = u_1 - 0.96u_3 + 0.91u_5$$

where f_1 is the force action into the first flexible mode. For simplicity, we use

$$u^TRu = (u_1 + u_3 + u_5)^2 + (u_1 - u_5)^2 + (u_1 - u_3 + u_5)^2$$

$$= u^T \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} u + u^T \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} u$$

$$+ u^T \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} u = u^T \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} u$$

In this way, the LQ design somewhat mimics the philosophy used in the previous design. As with the previous design, this may force energy into other modes.

We obtain state variable feedback controllers of the form of

$$u_1 = h_1^T x \tag{28a}$$

$$u_3 = h_3^T x \tag{28b}$$

$$u_5 = h_5^T x \tag{28c}$$

with

$$h_1^T = [\;9.85\;\; -89.11\;\; -10.44\;\; -135.27\;\; 1.12\;\; 57.83\;\; 0.15\;\; 0.83\;\; -0.10\;\; -4.61\;\; -0.01\;\; 2.00] \tag{29a}$$

$$h_3^T = [-10.10\;\; 130.74\;\; 21.40\;\; 29.57\;\; -10.10\;\; -165.95\;\; -0.09\;\; -2.58\;\; 0.28\;\; 7.38\;\; -0.10\;\; -4.89] \tag{29b}$$

$$h_5^T = [\;1.84\;\; -51.09\;\; -9.76\;\; 89.78\;\; 8.60\;\; 97.98\;\; -0.00\;\; 1.74\;\; -0.08\;\; -2.79\;\; 0.14\;\; 2.88] \tag{29c}$$

which yields a controlled system with closed-loop eigenvalues:

$$\lambda_1 = -21.5 \tag{30a}$$

$$\lambda_2 = -24.7 \tag{30b}$$

$$\lambda_3 = -24.8 \tag{30c}$$

$$\lambda_4 = -40.0 \tag{30d}$$

$$\lambda_{5,6} = -29.3 \pm 96.6i \tag{30e}$$

$$\lambda_{7,8} = -39.8 \pm 814.4i \tag{30f}$$

$$\lambda_{9,10} = -38.4 \pm 1163.6i \tag{30g}$$

$$\lambda_{11,12} = -38.5 \pm 1428.5i \tag{30h}$$

Figures 7a–7c illustrate the response of the two-element UA beam under unsaturated state variable feedback of the form of Eqs. (28), and Figs. 8a–8c illustrate the response saturated state variable feedback control of the form

$$u_i = \begin{Bmatrix} -6.2 & \text{if} & -h_i^T x < 6.2 \\ -h_i^T x & \text{if} & |h_i^T x| \le 6.2 \\ +6.2 & \text{if} & -h_i^T x > 6.2 \end{Bmatrix}, \quad i = 1,3,5 \tag{31}$$

where the saturation limit of 6.2 is the maximum force used in Eq. (25). Under this latter control, adequate positioning and damping of the system is obtained.

Figures 9a–9b illustrate the response of the four-element UA beam under the two-element control law given by Eq. (31). It is clear that, in this case, the UA beam is unstable. This result illustrates that state variable feedback under saturation is very model dependent and must be applied with utmost caution.

Discussion

It would appear that there are some distinct advantages to the open- and closed-loop design approach proposed here. Contrary to conventional wisdom, the open-loop portion of the control leads to a more robust design than possible through an overall closed-loop control design. This is due to two main features of the beam-positioning problem. The first feature is associated with the approximate nature of any finite model for the beam, and the second feature is associated with the finite control action available through the actuators. Since the positioning portion of the control is to be fast, the actuators will always be operating near maximum control. Control saturation is shown to have a destabilizing effect on a conventional linear system/quadratic cost design, particularly when this design is based on a lower-order model and then applied to a higher-order model.

An additional advantage of the open-loop portion of the control is that it allows, the suppression of selected flexible body modes through the use of redundant controllers. This can effectively eliminate much of the internal energy imparted to the beam, which would otherwise occur during the positioning phase. Since no state information is required for the open-loop controller, all of the difficulties associated with a full state feedback design are avoided.

The open-loop positioning phase is followed by a closed-loop damping phase. This is mainly an energy-dissipation phase: thus, the closed-loop design can take advantage of this fact. Velocity feedback at each actuator, with some limited position feedback to correct for any errors in the open-loop phase, appears to be adequate.

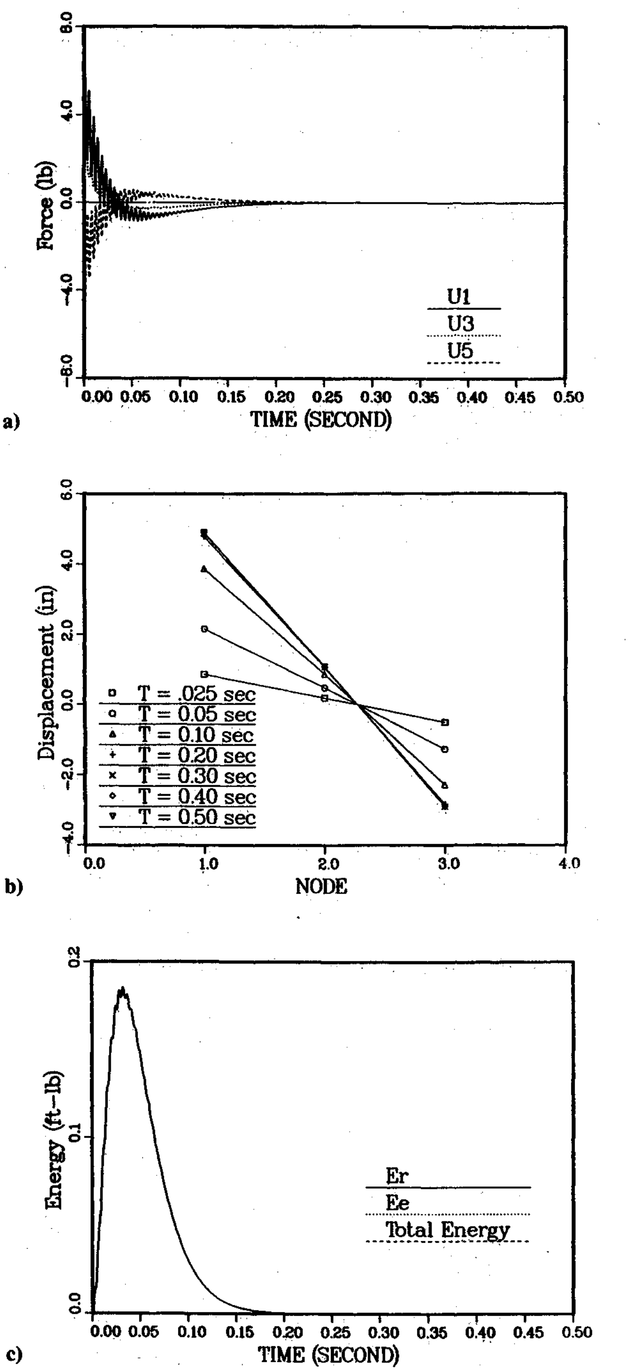

Fig. 7 Closed-loop positioning and damping of two-element beam using an LQ design with no control constraints ($b_1 = b_3 = b_5 = 1$).

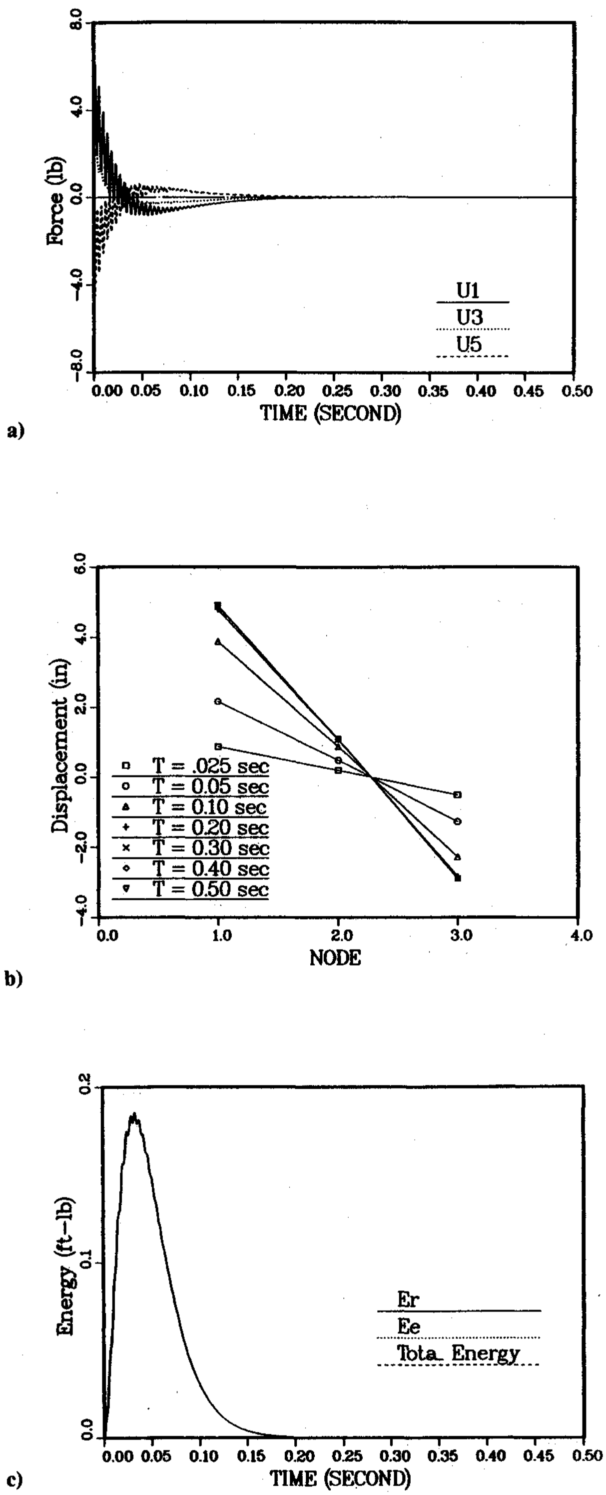

Fig. 8 Same as Fig. 7 except with bounded control.

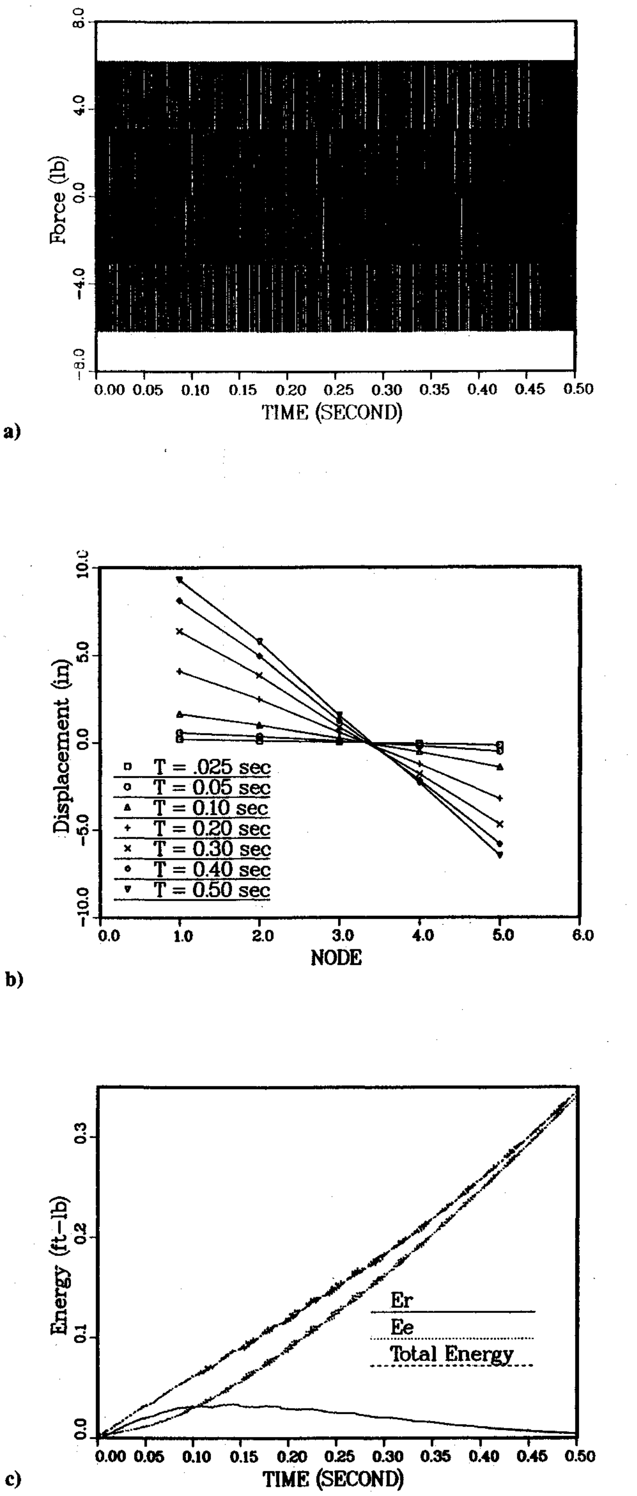

Fig. 9 Response of four-element beam using the two-element LQ design of Fig. 8.

Appendix

The mass and stiffness components for the UA beam with four equal elements are:

$m_1 = 0.0535 \times 10^{-2}$ $m_3 = 0.1070 \times 10^{-2}$ $m_5 = 0.1070 \times 10^{-2}$ $m_7 = 0.1070 \times 10^{-2}$ $m_9 = 0.0535 \times 10^{-2}$

$m_2 = 0.0322 \times 10^{-2}$ $m_4 = 0.0645 \times 10^{-2}$ $m_6 = 0.0645 \times 10^{-2}$ $m_8 = 0.0654 \times 10^{-2}$ $m_{10} = 0.0322 \times 10^{-2}$

$$
K = \begin{bmatrix}
163 & -917 & -163 & -917 & 0 & 0 & 0 & 0 & 0 & 0 \\
-917 & 6875 & 917 & 3438 & 0 & 0 & 0 & 0 & 0 & 0 \\
-163 & 917 & 326 & 0 & -163 & -917 & 0 & 0 & 0 & 0 \\
-917 & 3438 & 0 & 13750 & 917 & 3438 & 0 & 0 & 0 & 0 \\
0 & 0 & -163 & 917 & 326 & 0 & -163 & -917 & 0 & 0 \\
0 & 0 & -917 & 3438 & 0 & 13750 & 917 & 3438 & 0 & 0 \\
0 & 0 & 0 & 0 & -163 & 917 & 326 & 0 & -163 & -917 \\
0 & 0 & 0 & 0 & -917 & 3438 & 0 & 13750 & 917 & 3438 \\
0 & 0 & 0 & 0 & 0 & 0 & -163 & 917 & 163 & 917 \\
0 & 0 & 0 & 0 & 0 & 0 & -917 & 3438 & 917 & 6875
\end{bmatrix}
$$

$$
\Phi^T = \begin{bmatrix}
1.0000 & 0.0000 & 1.0000 & 0.0000 & 1.0000 & 0.0000 & 1.0000 & 0.0000 & 1.0000 & 0.0000 \\
1.0000 & 0.0444 & 0.5000 & 0.0444 & 0.0000 & 0.0444 & -0.5000 & 0.0444 & -1.0000 & 0.0444 \\
1.0000 & 0.1140 & -0.1650 & 0.0829 & -0.6699 & 0.0000 & -0.1650 & -0.0829 & 1.0000 & -0.1140 \\
0.9952 & 0.2451 & -1.0000 & 0.0444 & 0.0000 & -0.1564 & 1.0000 & 0.0444 & -0.9952 & 0.2451 \\
0.4292 & 0.2009 & -0.7146 & -0.0898 & 1.0000 & 0.0000 & -0.7146 & 0.0898 & 0.4292 & -0.2009 \\
0.0000 & -1.0000 & 0.0000 & 1.0000 & 0.0000 & -1.0000 & 0.0000 & 1.0000 & 0.0000 & -1.0000 \\
-0.0371 & 1.0000 & 0.0641 & -0.7156 & -0.0911 & 0.0000 & 0.0641 & 0.7156 & -0.0371 & -1.0000 \\
0.0788 & -0.9783 & -0.0799 & 0.0109 & 0.0000 & 1.0000 & 0.0799 & 0.0109 & -0.0788 & -0.9783 \\
0.1014 & -1.0000 & -0.0300 & -0.7031 & -0.0414 & 0.0000 & -0.0300 & 0.7031 & 0.1014 & 1.0000 \\
-0.1069 & 1.0000 & 0.0013 & 0.9856 & 0.0000 & 0.9712 & -0.0013 & 0.9856 & 0.1069 & 1.0000
\end{bmatrix}
$$

$$
\bar{K} = 10^4 \times \begin{bmatrix}
0.0 & & & & & \cdots & & & & 0 \\
& 0.0 & & & & & & & & \\
& & 0.0029 & & & & & & & \\
\cdot & & & 0.0366 & & & & & & \cdot \\
& & & & 0.0883 & & & & & \\
\cdot & & & & & 2.7500 & & & & \cdot \\
\cdot & & & & & & 1.8517 & & & \cdot \\
& & & & & & & 2.7786 & & \\
& & & & & & & & 3.7868 & \\
0 & & & & & \cdots & & & & 8.0929
\end{bmatrix}
$$

$$
\bar{M} = 10^{-2} \times \begin{bmatrix}
0.4280 & & & & & \cdots & & & & 0 \\
& 0.1610 & & & & & & & & \\
& & 0.1626 & & & & & & & \\
\cdot & & & 0.3257 & & & & & & \cdot \\
& & & & 0.2396 & & & & & \\
\cdot & & & & & 0.2580 & & & & \cdot \\
\cdot & & & & & & 0.1324 & & & \cdot \\
& & & & & & & 0.1282 & & \\
& & & & & & & & 0.1271 & \\
0 & & & & & \cdots & & & & 0.2518
\end{bmatrix}
$$

The mass and stiffness components for the UA beam with two unequal elements are:

$m_1 = 0.1046 \times 10^{-2}$ $m_3 = 0.2140 \times 10^{-2}$ $m_5 = 0.1094 \times 10^{-2}$

$m_2 = 0.2411 \times 10^{-2}$ $m_4 = 0.5166 \times 10^{-2}$ $m_6 = 0.2755 \times 10^{-2}$

$$K = \begin{bmatrix} 21.8 & -239.7 & -21.8 & -239.7 & 0 & 0 \\ -239.7 & 3515.6 & 239.7 & 1757.8 & 0 & 0 \\ -21.8 & 239.7 & 40.9 & 20.4 & -19.1 & -219.3 \\ -239.7 & 1757.8 & 20.4 & 6878.4 & 219.3 & 1681.4 \\ 0 & 0 & -19.1 & 219.3 & 19.1 & 219.3 \\ 0 & 0 & -219.3 & 1681.4 & 219.3 & 3362.8 \end{bmatrix}$$

$$\Phi^T = \begin{bmatrix} 1.00000 & 0.00085 & 0.98133 & 0.00085 & 0.96182 & 0.00085 \\ 0.98053 & 0.04401 & 0.01227 & 0.04401 & -1.00000 & 0.04401 \\ 1.00000 & 0.13136 & -0.95651 & 0.00576 & 0.91492 & -0.12589 \\ 0.02567 & 0.85014 & -0.02563 & -0.91979 & 0.02560 & 1.00000 \\ 0.13512 & -0.94688 & -0.15720 & 0.17497 & 0.17832 & 1.00000 \\ -0.20083 & 1.00000 & 0.02998 & 0.76771 & 0.13344 & 0.58130 \end{bmatrix}$$

Acknowledgments

Research was supported by Honeywell, Sperry Space Systems Division. The authors are indebted to Dave Cunningham and Doug Havenhell of Sperry Space Systems for their comments and suggestions related to this work.

References

[1]Vincent, T. L., Joshi, S. P., and Lin, Y. C., "Positioning and Active Damping of Spring-Mass Systems," *Journal of Dynamic Systems, Measurement and Control,* Vol. 3, No. 4, Dec. 1989, pp. 592–599.

[2]Horner, G. C., and Walz, J. E., "A Design Technique for Determining Actuator Gains in Spacecraft Vibration Control," AIAA Paper 85-0628, April 1985.

[3]Medanic, J., Tharp, H. S., and Perkins, W. R., "Pole-Placement by Performance Criterion Modification," *IEEE Transactions on Automatic Control,* Vol. 33, No. 5, May 1988, pp. 479–483.

[4]Hanagud, S., Obal, M. W., and Calise, A. J., "Optimal Vibration Control by the Use of Piezoceramic Sensors and Actuators," AIAA Paper 87-0959, April 1987.

[5]Venkayya, V. B., and Tischler, V. A., "Frequency Control and the Effect on the Dynamic Response of Flexible Structures," AIAA Paper 84-1044, May 1984.

[6]Kailath, T., *Linear Systems,* Prentice-Hall, Englewood Cliffs, NJ, 1980.

[7]Chen, C. T., *Linear System Theory and Design,* Holt, Rinehart, & Winston, New York, 1984.

[8]Balas, M. J., "Direct Velocity Feedback Control of Large Space Structures," *Journal of Guidance and Control,* Vol. 2, No. 2, 1979, pp. 252–253.

[9]Schäfer, B. E., and Holzach, H., "Experimental Research on Flexible Beam Modal Control," *Journal of Guidance and Control,* Vol. 8, No. 5, 1985, pp. 597–604.

[10]Bennighof, J. K., and Meirovitch, L., "Implementation of Active Suppression of Traveling Waves in Structures," AIAA Paper 87-0942, April 1987.

[11]Meirovitch, L., and Baruh, H., "Nonlinear Natural Control of an Experimental Beam," *Journal of Guidance and Control,* Vol. 7, No. 4, 1984, pp. 437–442.

[12]Greenwood, D. T., *Principles of Dynamics,* Prentice-Hall, Englewood Cliffs, NJ, 1988.

[13]Bathe, K. J., *Finite Element Procedures in Engineering Analysis,* Prentice-Hall, Englewood Cliffs, NJ, 1982.

[14]Dawe, D. J., *Matrix and Finite Element Displacement Analysis of Structures,* Oxford Univ. Press, New York, 1984.

[15]Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijthoff and Noordhoff, Leiden, the Neitherlands, 1980.

[16]Meirovitch, L., and Baruh, H., "The Implementation of Modal Filters for Control of Structures," *Journal of Guidance and Control,* Vol. 8, No. 6, 1985, pp. 707–716.

[17]Meirovitch, L., *Elements of Vibration Analysis,* McGraw-Hill, New York, 1975.

[18]Leitmann, G., *The Calculus of Variations and Optimal Control,* Plenum, New York, 1981.

[19]Athans, M., and Falb, P. L., *Optimal Control,* McGraw-Hill, New York, 1966.

[20]Joshi, S. P., Vincent, T. L., and Lin, Y. C., "Control for Energy Dissipation in Structures," *Journal of Guidance, Control, and Dynamics* (to be published).